CFSYSTEMS

Precise Frequency and Amplitude Tracking of Waveforms

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Note

This document has been assembled from the major parts of three documents in which the Precise Signal Component method was originally derived. The original document was assigned the number CFS-175 and other parts, including the two assembled here, were assigned CFS-175 with some supplementary qualifier. A later, considerably abridged version was prepared for use in the patent application and was assigned the number CFS-185.

The first part of this document gives the basic considerations for Precise Signal Component and derives the basic form for a single frequency component. The second part, Platform Least Squares, derives a surrogate function to reduce or eliminate confounding between widely spaced frequencies. The third part, Multiple Adjacent Frequencies, derives the means of handling multiple components having frequencies that are closely spaced.

This material is identical with the method described in USP 6,751,564, but expands upon it considerably.

This document is viewable only. I believe this will be adequate for people who do not intend to study it. Please contact me through our web site if you need a printable version. I am aware that the no-print can be defeated, but again I ask that you contact me instead. I really need to know if and how people are finding these documents useful, and this seems one of the few ways I have to encourage feedback.

The Fourier Transform and the DFT/FFT

The preeminent tool of frequency spectrum analysis is the Fourier transform, typically implemented in the form of the Fast Fourier Transform (FFT) or another variety of Digital Fourier Transform (DFT). The Fourier transform of a signal $g(t)$ is defined as:

$$\mathfrak{F}(g(t), f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i f t}dt \tag{1}$$

For those used to thinking of frequency spectrums in terms of sines and cosines, it is helpful to recall that $e^{ix} = cos(x) + i\, sin(x)$. Thus the transform can also be expressed in the form:

$$\mathfrak{F}(g(t), f) = \int_{-\infty}^{\infty} g(t)cos(2\pi i f t)dt - i\int_{-\infty}^{\infty} g(t)sin(2\pi i f t)dt$$

However, it is often easier to work with the complex exponential form. The infinite Fourier transform cannot be used directly in data analysis. We are forced to examine – sample – signals over a finite time interval. Moreover, the time interval usually must be quite short, as few signals in the real world remain of constant character for long periods of time. Thus the finite Fourier transform is evaluated only over a finite sample time τ.

$$\mathfrak{F}(g(t), n) = \frac{1}{\tau}\int_{0}^{\tau} g(t)e^{-\frac{i2\pi nt}{\tau}}dt \tag{2}$$

Note that the frequency f has been replaced by n/τ. In practice, n is taken as an integer and each $e^{-\frac{i2\pi nt}{\tau}}$ is periodic (see the sine and cosine form) over the sample interval, τ. Over τ, M evenly spaced samples of $g(t)$ are taken, at $t = 0$, τ/M, $2\tau/M$, ..., $(M-1)\tau/M$. The integral is evaluated for frequency numbers $n = 0, ..., M-1$. Note in passing that the standard summation form has the interesting feature that the values start at zero but stop short of τ. Because the functions $e^{-\frac{i2\pi nt}{\tau}}$ are orthogonal, this approach is robust, and the integral is almost always approximated directly as the corresponding sum, $\frac{1}{M}\sum_{k=0}^{M-1} g(k\tau/m)e^{-\frac{i2\pi nk}{M}}$, evaluated for each frequency number, $n = 0, ..., M-1$. Orthoganality – linear independence – guarantees that in evaluating this amplitude-measuring integral at one frequency number there will be no interference from any of the other frequency components being measured.

Current Approaches to the Base Problem

Fourier analysis is a powerful and easily applied tool. There have been many approaches to modifying the DFT approach in order to ameliorate the problems described above. These have principally been of three varieties. First, *windowing*, where the data function $g(t)$ is pre-multiplied by another function carefully chosen to reduce the influence of long sample times and changing properties of $g(t)$ within the interval. While used for this purpose, windowing has also seen extensive use in minimizing the $sin(x)/x$ "leakage" or "noise" associated with the Fourier transform, as described later.

The second major category of modifications to the DFT for sounds analysis has been partitioning, where the spectrum is divided into several ranges and each range treated differently using separate DFTs. This method allows coarser DFTs with shorter sample intervals in the high frequency ranges.

The third major category of modifications is the "traveling DFT," where the DFT is successively applied to the data, discarding a few points at the beginning of the sample interval and adding a corresponding number of new points to advance the sample interval along the timeline. The result is then analyzed using knowledge gained from experience with exactly how the DFT will change when passing through various areas of changing signals.

While these modifications are useful and may indeed be used in some form to augment the method described in this document, they do not in themselves provide solid solutions to the base problem of the DFT, which is that the resolution of the frequency and amplitude of component signals to a degree consistent with human hearing requires sample times that invalidate the constant signal assumption basic to the DFT.

The Root Assumption

In the following, we develop a means of using the tools of standard Fourier analysis whereby we are not trapped by the limitations. To do this requires that one simple assumption be made:

At any instant, the signal which we wish to analyze is comprised of a small number of specific frequency components.

By "small number" we mean less than infinite and in general, less than the number of standard Fourier coefficients that will be computed as part of the method. By "specific frequency components," we mean that the frequencies of the actual signals will not normally correspond exactly to the numbered frequencies of any Fourier analysis or DFT that is being used. By "at any instant" we mean that the frequency and amplitude of the specific frequency components may change with time. Such changes will usually be slow with respect to the specific frequencies themselves, but may also involve sudden transients.

Development of the Method

Returning to Equation (2) from above:

$$\mathfrak{F}(g(t),n) = \frac{1}{\tau}\int_0^\tau g(t)e^{-\frac{i2\pi nt}{\tau}}dt \qquad [2]$$

Suppose $g(t)$ is a sinusoid of frequency that does not match the integer spacings $2n\pi/\tau$, but is rather $2(n+\delta)\pi/\tau$. Then $g(t) = Ae^{\frac{i2\pi(n+\delta)t}{\tau}}$, where A is a complex constant and

$$\mathfrak{F}(g(t),n) = \frac{A}{\tau}\int_0^\tau e^{\frac{i2\pi(n+\delta)t}{\tau}}e^{-\frac{i2\pi nt}{\tau}}dt$$

$$= \frac{A}{\tau}\int_0^\tau e^{\frac{i2\delta\pi t}{\tau}}dt \qquad (8)$$

Which evaluates to:

$$\mathfrak{F}(g(t),n) = \frac{iA\left(1-e^{i2\pi\delta}\right)}{2\pi\delta} \qquad (9)$$

The magnitude of this evaluates to:

$$\sqrt{\mathfrak{F}(g(t),n)\overline{\mathfrak{F}(g(t),n)}} = \left|\frac{|A|\,sin(\pi\delta)}{\pi\delta}\right| \qquad (10)$$

Where the overbar indicates complex conjugate. This is the $\frac{sin(x)}{x}$ "sideband" so commonly associated with the use of the FFT. Notice that the value is independent of n and relates only to δ, the displacement from the frequency of interest, n. Common practice is to regard these sidebands as an unwanted form of "leakage" or "noise" associated with the method and to attack the problem of the sidebands by carefully choosing a "windowing" function, which when multiplied into $g(t)$ prior to the transform will cause the sidebands to be as small as possible. In actuality as we shall see below, the information in these sidebands is a key element of full frequency analysis. Windowing typically obscures or destroys this necessary information. Equation (10) shows that a reason for "sideband leakage" is that the signal being measured does not coincide precisely with any of the DFT reference frequencies.

Let us examine Equation (9) more carefully. Note that δ can be any value. It is useful at this point to define $\delta = k + \epsilon$, where k is a principle integer value of δ and ϵ is the difference remaining, typically but not necessarily a fraction.

Thus we have:

$$\mathfrak{F}(g(t), n) = \frac{iA\left(1 - e^{i2\pi(k+\epsilon)}\right)}{2\pi(k+\epsilon)} = \frac{iA\left(1 - e^{i2\pi k} e^{i2\pi\epsilon}\right)}{2\pi(k+\epsilon)}$$

but k is an integer value and therefore $e^{i2\pi k} = 1$, so

$$\mathfrak{F}(g(t), n) = \frac{iA(1 - e^{i2\pi\epsilon})}{2\pi(k+\epsilon)} \tag{11}$$

In practice we will be analyzing a set of values of $\mathfrak{F}(n)$ for a sequential range of n. {Note: 7/2004: The following is more tricky than it looks. We are defining a function which exists only for integer k but succeed in using it as a continuous function. This is nearly as fortuitous for this application as the FFT is for standard Fourier series analysis.} For each of the successive increasing values of n, δ will be one less, since the (fixed) frequency of our $g(t)$ is represented $n + \delta$. Since $\delta = k + \epsilon$, k will also increase by one for each successive n and ϵ will remain constant. Thus, from a sequence of data y, where we use the shortened notation $y \equiv \mathfrak{F}(g(t), n)$, we will be trying to determine ϵ from the values of y at different values of k. The actual function we will be trying to fit to the data will be

$$y = A\frac{i(1 - e^{i2\pi\epsilon})}{2\pi(k+\epsilon)} \tag{12}$$

For this analysis, we can recognize that $A' = A\frac{i(1-e^{i2\pi\epsilon})}{2\pi}$ is a single complex constant for the purposes of the fit and ϵ is a single real constant. The value of A can be found once A' and ϵ have been determined. This leaves us with

$$y = \frac{A'}{(k + \epsilon)} \tag{13}$$

Equation (13) makes the remarkable statement that the $\frac{sin(x)}{x}$ "sideband leakage" due to actual signal frequencies not matching DFT reference frequencies produces coefficients which correlate in a very simple way – a linearly biased inverse – with the degree to which the actual frequency mismatches the nearest DFT frequencies.

It is very desirable to put fitting equations into linear form *(linear with respect to the fitting constants)* if possible, as it greatly simplifies the computations required to fit the equations.

$$k = -\epsilon + \frac{A'}{y} \tag{14}$$

Equation (14) is one such form. Equation (14) is unusual for two reasons. First, a least squares fit to this form will be a fit to k, normally thought of as the independent variable in terms of y, normally thought of as the dependent variable. Second, k and ϵ are definitely real, while A' and y are complex. Special consideration will be given both these unusual features.

For the least squares fit to Equation (14), normally the sum

$$\sum_j \left(k_j - \widehat{k_j}\right)\overline{\left(k_j - \widehat{k_j}\right)}$$

would be minimized, where $\widehat{k_j}$ is the value calculated from the fit that corresponds to k_j. Here the overbar indicates complex conjugate, and we must sum the magnitude differences squared instead of $\left(k_j - \widehat{k_j}\right)^2$ because $\widehat{k_j}$ must be regarded as complex, containing the term $\frac{A'}{y_j}$. Because k_j is real, the $\widehat{k_j}$ resulting from the fit should be very nearly real. However, ordinarily we would be minimizing $\sum_j (y_j - \widehat{y_j})\overline{(y_j - \widehat{y_j})}$, the sum of the magnitude differences squared for the independent variable in Equation (13). We can approximate this by including a weighting factor w_j in the original sum, such that:

$$w_j\left(k_j - \widehat{k_j}\right)\overline{\left(k_j - \widehat{k_j}\right)} \approx (y_j - \widehat{y_j})\overline{(y_j - \widehat{y_j})}$$

To do this in this case it is useful to think of this in terms of small differences, whereby we can approximate:

$$w_j|\Delta k_j|^2 \approx |\Delta y_j|^2$$

$$w_j \approx \frac{|\Delta y_j|^2}{|\Delta k_j|^2} \approx \left|\frac{dy}{dk}\right|^2 = \frac{|y_j|^4}{|A'|^2}$$

with $\frac{dy}{dk}$ resulting from differentiating Equation (13) [or Equation (14)]. Since $|A'|^2$ is a constant, it will have no effect on the minimization, so the primary weighting factor is:

$$w_j = |y_j|^4$$

Note that this weighting factor merely restores the fit so that it more closely matches a least squares fit to Equation (13). Other weighting factors may be found to have more desirable characteristics in practice.

So, we wish to minimize:

$$\sum_j w_j \left(k_j - \widehat{k_j}\right)\overline{\left(k_j - \widehat{k_j}\right)}$$

$$\sum_j w_j \left(k_j + \epsilon - \frac{A'}{y_j}\right)\overline{\left(k_j + \epsilon - \frac{A'}{y_j}\right)}$$

$$\sum_j w_j \left(k_j + \epsilon - \frac{A'}{y_j}\right)\left(k_j + \epsilon - \frac{\overline{A'}}{\overline{y_j}}\right)$$

with respect to the values of the constants real ϵ and complex A'. Here it is useful to write $A' = A'_R + iA'_I$

$$\sum_j w_j \left(k_j + \epsilon - \frac{A'_R}{y_j} - i\frac{A'_I}{y_j}\right)\left(k_j + \epsilon - \frac{A'_R}{\overline{y_j}} + i\frac{A'_I}{\overline{y_j}}\right)$$

To find the minimum, we set the derivatives of the sum equal to zero, for each of ϵ, A'_R, and A'_I.

$$\frac{\partial}{\partial \epsilon}\sum_j w_j \left(k_j + \epsilon - \frac{A'_R}{y_j} - i\frac{A'_I}{y_j}\right)\left(k_j + \epsilon - \frac{A'_R}{\overline{y_j}} + i\frac{A'_I}{\overline{y_j}}\right) = 0$$

$$\sum_j w_j \left(k_j + \epsilon - \frac{A'_R}{y_j} - i\frac{A'_I}{y_j} + k_j + \epsilon - \frac{A'_R}{\overline{y_j}} + i\frac{A'_I}{\overline{y_j}}\right) = 0$$

$$\sum_j w_j \left(2k_j + 2\epsilon - A'_R \left(\frac{1}{y_j} + \frac{1}{\overline{y_j}} \right) - A'_I i \left(\frac{1}{y_j} - \frac{1}{\overline{y_j}} \right) \right) = 0$$

$$\sum_j w_j \left(2k_j + 2\epsilon - A'_R \left(\frac{\overline{y_j} + y_j}{y_j \overline{y_j}} \right) - A'_I i \left(\frac{\overline{y_j} - y_j}{y_j \overline{y_j}} \right) \right) = 0$$

$$\sum_j w_j \left(k_j + \epsilon - A'_R \frac{Re(y_j)}{|y_j|^2} - A'_I \frac{Im(y_j)}{|y_j|^2} \right) = 0$$

where $|y_j|^2$ denotes the squared magnitude of y_j.

$$\frac{\partial}{\partial A'_R} \sum_j w_j \left(k_j + \epsilon - \frac{A'_R}{y_j} - i \frac{A'_I}{y_j} \right) \left(k_j + \epsilon - \frac{A'_R}{\overline{y_j}} + i \frac{A'_I}{\overline{y_j}} \right) = 0$$

$$-\sum_j w_j \left(\frac{k_j}{\overline{y_j}} + \epsilon \frac{1}{\overline{y_j}} - \frac{A'_R}{y_j \overline{y_j}} + \frac{k_j}{y_j} + \epsilon \frac{1}{y_j} - \frac{A'_R}{y_j \overline{y_j}} \right) = 0$$

$$\sum_j w_j \left(k_j \left(\frac{1}{\overline{y_j}} + \frac{1}{y_j} \right) + \epsilon \left(\frac{1}{\overline{y_j}} + \frac{1}{y_j} \right) - 2 \frac{A'_R}{y_j \overline{y_j}} \right) = 0$$

$$\sum_j w_j \left(k_j \left(\frac{y_j + \overline{y_j}}{\overline{y_j} y_j} \right) + \epsilon \left(\frac{y_j + \overline{y_j}}{\overline{y_j} y_j} \right) - 2 \frac{A'_R}{y_j \overline{y_j}} \right) = 0$$

$$\sum_j w_j \left(k_j \frac{Re(y_j)}{|y_j|^2} + \epsilon \frac{Re(y_j)}{|y_j|^2} - \frac{A'_R}{|y_j|^2} \right) = 0$$

$$\frac{\partial}{\partial A'_I} \sum_j w_j \left(k_j + \epsilon - \frac{A'_R}{y_j} - i \frac{A'_I}{y_j} \right) \left(k_j + \epsilon - \frac{A'_R}{\overline{y_j}} + i \frac{A'_I}{\overline{y_j}} \right) = 0$$

$$\sum_j w_j \left(ik_j \left(\frac{1}{\overline{y_j}} - \frac{1}{y_j} \right) + i\epsilon \left(\frac{1}{\overline{y_j}} - \frac{1}{y_j} \right) + 2\frac{A'_I}{y_j \overline{y_j}} \right) = 0$$

$$\sum_j w_j \left(ik_j \left(\frac{y_j - \overline{y_j}}{y_j \overline{y_j}} \right) + i\epsilon \left(\frac{y_j - \overline{y_j}}{y_j \overline{y_j}} \right) + 2\frac{A'_I}{y_j \overline{y_j}} \right) = 0$$

$$\sum_j w_j \left(-k_j \frac{Im(y_j)}{|y_j|^2} - \epsilon \frac{Im(y_j)}{|y_j|^2} + \frac{A'_I}{|y_j|^2} \right) = 0$$

The three equations are:

$$\sum_j w_j \left(k_j + \epsilon - A'_R \frac{Re(y_j)}{|y_j|^2} - A'_I \frac{Im(y_j)}{|y_j|^2} \right) = 0$$

$$\sum_j w_j \left(k_j \frac{Re(y_j)}{|y_j|^2} + \epsilon \frac{Re(y_j)}{|y_j|^2} - \frac{A'_R}{|y_j|^2} \right) = 0$$

$$\sum_j w_j \left(-k_j \frac{Im(y_j)}{|y_j|^2} - \epsilon \frac{Im(y_j)}{|y_j|^2} + \frac{A'_I}{|y_j|^2} \right) = 0$$

Partitioning out the sums:

$$-\epsilon \sum_j w_j + A'_R \sum_j w_j \frac{Re(y_j)}{|y_j|^2} + A'_I \sum_j w_j \frac{Im(y_j)}{|y_j|^2} = \sum_j w_j k_j$$

$$-\epsilon \sum_j w_j \frac{Re(y_j)}{|y_j|^2} + A'_R \sum_j w_j \frac{1}{|y_j|^2} = \sum_j w_j k_j \frac{Re(y_j)}{|y_j|^2}$$

$$-\epsilon \sum_j w_j \frac{Im(y_j)}{|y_j|^2} + A'_I \sum_j w_j \frac{1}{|y_j|^2} = \sum_j w_j k_j \frac{Im(y_j)}{|y_j|^2}$$

In matrix form:

$$\begin{pmatrix} -\sum_j w_j & \sum_j w_j \frac{Re(y_j)}{|y_j|^2} & \sum_j w_j \frac{Im(y_j)}{|y_j|^2} \\ -\sum_j w_j \frac{Re(y_j)}{|y_j|^2} & \sum_j w_j \frac{1}{|y_j|^2} & 0 \\ -\sum_j w_j \frac{Im(y_j)}{|y_j|^2} & 0 & \sum_j w_j \frac{1}{|y_j|^2} \end{pmatrix} \begin{pmatrix} \epsilon \\ A'_R \\ A'_I \end{pmatrix} = \begin{pmatrix} \sum_j w_j k_j \\ \sum_j w_j k_j \frac{Re(y_j)}{|y_j|^2} \\ \sum_j w_j k_j \frac{Im(y_j)}{|y_j|^2} \end{pmatrix} \quad (15)$$

And from this the three coefficients ϵ, A'_R, and A'_I can be found, giving ϵ and A'. We can also calculate A that appears in Equation (13) as $A = -2\pi i A'/(1 - e^{i2\pi\epsilon})$. This gives both the precise frequency of the tone and its precise amplitude.

If we use the principle weighting factor $w_j = |y_j|^4$ we obtain:

$$\begin{pmatrix} -\sum_j |y_j|^4 & \sum_j |y_j|^2 Re(y_j) & \sum_j |y_j|^2 Im(y_j) \\ -\sum_j |y_j|^2 Re(y_j) & \sum_j |y_j|^2 & 0 \\ -\sum_j |y_j|^2 Im(y_j) & 0 & \sum_j |y_j|^2 \end{pmatrix} \begin{pmatrix} \epsilon \\ A'_R \\ A'_I \end{pmatrix} = \begin{pmatrix} \sum_j |y_j|^4 k_j \\ \sum_j |y_j|^2 Re(y_j) k_j \\ \sum_j |y_j|^2 Im(y_j) k_j \end{pmatrix} \quad (16)$$

Both of these matrices are of the form:

$$\begin{pmatrix} -a & b & d \\ -b & c & 0 \\ -d & 0 & c \end{pmatrix} \begin{pmatrix} \epsilon \\ A'_R \\ A'_I \end{pmatrix} = \begin{pmatrix} Y1 \\ Y2 \\ Y3 \end{pmatrix}$$

Using Maple V Version 5, it is easy to obtain the algebraic inverse of the 3x3 matrix so that:

$$\begin{pmatrix} \epsilon \\ A'_R \\ A'_I \end{pmatrix} = \frac{1}{ac - b^2 - d^2} \begin{pmatrix} -c & b & d \\ -b & \frac{ac-d^2}{c} & \frac{bd}{c} \\ -d & \frac{bd}{c} & \frac{ac-b^2}{c} \end{pmatrix} \begin{pmatrix} Y1 \\ Y2 \\ Y3 \end{pmatrix}$$

$$D = ac - b^2 - d^2$$

$$\epsilon = \frac{(-cY1 + bY2 + dY3)}{D}$$

$$A'_R = \frac{\left(- bY1 + \frac{ac-d^2}{c}Y2 + \frac{bd}{c}Y3 \right)}{D}$$

$$A'_I = \frac{\left(- dY1 + \frac{bd}{c}Y2 + \frac{ac-b^2}{c}Y3 \right)}{D}$$

From Equation (15),

$$D = \sum_j w_j \sum_j w_j \frac{1}{|y_j|^2} - \left(\sum_j w_j \frac{Re(y_j)}{|y_j|^2} \right)^2 - \left(\sum_j w_j \frac{Im(y_j)}{|y_j|^2} \right)^2$$

$$a = \sum_j w_j$$

$$b = \sum_j w_j \frac{Re(y_j)}{|y_j|^2}$$

$$c = \sum_j w_j \frac{1}{|y_j|^2}$$

$$d = \sum_j w_j \frac{Im(y_j)}{|y_j|^2}$$

$$Y1 = \sum_j w_j k_j$$

$$Y2 = \sum_j w_j k_j \frac{Re(y_j)}{|y_j|^2}$$

$$Y3 = \sum_j w_j k_j \frac{Im(y_j)}{|y_j|^2}$$

And for $w_j = |y_j|^4$, from Equation (16)

$$D = \sum_j |y_j|^4 \sum_j |y_j|^2 - \left(\sum_j |y_j|^2 Re(y_j) \right)^2 - \left(\sum_j |y_j|^2 Im(y_j) \right)^2$$

$$a = \sum_j |y_j|^4$$

$$b = \sum_j |y_j|^2 Re(y_j)$$

$$c = \sum_j |y_j|^2$$

$$d = \sum_j |y_j|^2 Im(y_j)$$

$$Y1 = \sum_j |y_j|^4 k_j$$

$$Y2 = \sum_j |y_j|^2 k_j Re(y_j)$$

$$Y3 = \sum_j |y_j|^2 k_j Im(y_j)$$

Platform Least Squares
The second major development in the method

Using the form derived above:

$$y = \frac{A'}{(k + \epsilon)} \qquad [13](1)$$

will produce reasonably accurate frequencies and amplitudes of generated mixed frequency signals using the method described in that report. For example, the following results were obtained for a randomly generated mixed frequency signal starting with a Fast Fourier Transform on 2048 (real) points.

F in	F out	Amplitude in		Amplitude out	
39.206	39.211	-58.813	-161.327	-55.886	-161.809
90.839	90.839	-368.906	65.213	-368.372	65.922
288.269	288.270	315.810	-295.113	316.009	-294.643
362.163	362.188	464.176	139.473	450.890	175.240
379.699	379.691	273.214	-407.479	263.194	-412.661
446.528	446.518	-128.046	9.567	-127.715	13.357
745.146	745.144	-272.562	431.675	-270.279	435.579
785.307	785.321	-374.069	425.528	-393.216	410.000
797.325	797.343	199.015	-216.649	229.429	-220.817
806.834	806.828	-393.595	-176.954	-397.911	-171.716
908.112	908.111	-187.935	-471.246	-188.234	-473.576
1014.719	1014.719	497.430	-27.095	496.677	-33.983

Table I
Sample Results of Original Least Squares Fit

{Note: 7/2004: Understand that the comments here were made in the course of developing the method and do not refer to it's current state of development.} Using the above method, the behavior shown here is typical for the majority of components in generated test frequency signals. However, the results for some components can show much greater errors, particularly in amplitude. A number of different phenomena contribute to such errors, and these are currently being studied. The method has not yet been tested using real-world data; there currently is no means of determining the precise frequencies and amplitudes in such data and thus no way of checking whether the method is actually working properly.

It should be noted that the first point in achieving accurate results regards the use of k. The temptation is to simply use the index of the array element in the FFT array as k. While mathematically sound, this approach is disastrous in practice for numerical reasons. When k is a large number, the solution of the least squares equations leads to the subtraction of nearly equal large numbers to obtain a difference that is a tiny fraction of the original operands. Thus it is important to reduce the size of k by subtracting a bias from it. It is convenient to use a bias so that the range of k either starts at 1 or straddles 0 if k is not used as a divisor in the calculation. This bias is added back into the frequency at the conclusion of the least squares analysis.

It is also clear that much can be gained by tuning the parameters of the least squares fit. This includes the number of points included in the fit, the determination of the approximate position of frequencies to be analyzed within the FFT array, the positioning of the range of points about that approximate frequency, and the weighting of the individual points within the least squares fit. The above results are essentially a first cut at this, but no real tuning has been performed at this point.

As shown above, Equation [13](1) is reasonably accurate but does suffer from the cross interference between frequency components. What happens in practice is that the full signal also contains the tails of all the other frequencies and these tend to form a platform under the single frequency signal. Thus a first approximation of what we have is

$$y = \frac{A'}{(k+\epsilon)} + C \tag{2}$$

where C is a complex constant representing the sum of all the tails of frequencies to either side. Of course C is not really constant and each signal that contributes to it will taper off from one side to the other, the direction of the taper depending on whether the interfering signal is above or below the frequency of the target signal. Since signals above the one of interest will generally taper opposite to those below, the sum will gradually twist going along the frequency interval being analyzed. Thus a better approximation would be

$$y = \frac{A'}{(k+\epsilon)} + C + Sk + Tk^2 \tag{3}$$

Multiplying out, we get

$$yk + \epsilon y = A' + Ck + Sk^2 + Tk^3 + C\epsilon + S\epsilon k + T\epsilon k^2 \tag{4}$$

dividing by k

$$y = A'\frac{1}{k} + C + Sk + Tk^2 + C\epsilon\frac{1}{k} + S\epsilon + T\epsilon k - \epsilon\frac{y}{k} \tag{5}$$

The variables are y and k, so collecting terms according to constants

$$y = (C + S\epsilon) + (A' + C\epsilon)\frac{1}{k} + (S + T\epsilon)k + Tk^2 - \epsilon\frac{y}{k} \tag{6}$$

This is in the linear form

$$y = a_0 + a_1\frac{1}{k} + a_2 k + a_3 k^2 + a_4\frac{y}{k} \tag{7}$$

where $a_0 ... a_3$ are complex and a_4, which corresponds to $-\epsilon$, is real only. When the constants $a_0 ... a_4$ are obtained, then

$$\epsilon = -a_4 \tag{8}$$

$$T = a_3 \tag{9}$$

$$S = a_2 - T\epsilon \tag{10}$$

$$C = a_0 - S\epsilon \tag{11}$$

$$A' = a_1 - C\epsilon \tag{12}$$

as before the original

$$A = -2\pi i A' / \left(1 - e^{i2\pi\epsilon}\right)$$

can be obtained from the result.

To perform a least squares fit on this form, we need to find the minimum of $\sum_j w_j (y_j - \widehat{y_j})\overline{(y_j - \widehat{y_j})}$=

$$\sum_j w_j \left(y_j - a_0 - a_1\frac{1}{k_j} - a_2 k_j - a_3 k_j^2 - a_4\frac{y_j}{k_j}\right)\overline{\left(y_j - a_0 - a_1\frac{1}{k_j} - a_2 k_j - a_3 k_j^2 - a_4\frac{y_j}{k_j}\right)} =$$

$$\sum_j w_j \left(y_j - a_0 - a_1\frac{1}{k_j} - a_2 k_j - a_3 k_j^2 - a_4\frac{y_j}{k_j}\right)\left(\overline{y_j} - \overline{a_0} - \overline{a_1}\frac{1}{k_j} - \overline{a_2} k_j - \overline{a_3} k_j^2 - a_4\frac{\overline{y_j}}{k_j}\right)$$

Here $a_0, ..., a_3,$ and y_j are the complex quantities, the remainder all being real variables and constants. Because a_4 *must* be maintained as a real, to correspond with ϵ, it is expedient to expand all the coefficients.

$$\sum_j w_j \left(y_j - a_{0R} - i\,a_{0I} - a_{1R}\tfrac{1}{k_j} - i\,a_{1I}\tfrac{1}{k_j} - a_{2R}k_j - i\,a_{2I}k_j - a_{3R}k_j^2 - i\,a_{3I}k_j^2 - a_4\tfrac{y_j}{k_j}\right)$$

$$\times \left(\overline{y_j} - a_{0R} + i\,a_{0I} - a_{1R}\tfrac{1}{k_j} + i\,a_{1I}\tfrac{1}{k_j} - a_{2R}k_j + i\,a_{2I}k_j - a_{3R}k_j^2 + i\,a_{3I}k_j^2 - a_4\tfrac{\overline{y_j}}{k_j}\right)$$

We must successively take the derivative of this sum with respect to $a_{0R}, a_{0I}, a_{1R}, a_{1I}, a_{2R}, a_{2I}, a_{3R}, a_{3I},$ and a_4, set each expression to zero, and solve to find the minimum.

$$\frac{\partial}{\partial a_{0R}}\sum_j w_j(...)(...) = 0 =$$
$$\sum_j w_j\Big(- y_j + a_{0R} + i\, a_{0I} + a_{1R}\tfrac{1}{k_j} + i\, a_{1I}\tfrac{1}{k_j} + a_{2R}k_j + i\, a_{2I}k_j + a_{3R}k_j^2 + i\, a_{3I}k_j^2 + a_4\tfrac{y_j}{k_j}\Big)$$
$$+ w_j\Big(- \overline{y_j} + a_{0R} - i\, a_{0I} + a_{1R}\tfrac{1}{k_j} - i\, a_{1I}\tfrac{1}{k_j} + a_{2R}k_j - i\, a_{2I}k_j + a_{3R}k_j^2 - i\, a_{3I}k_j^2 + a_4\tfrac{\overline{y_j}}{k_j}\Big)$$

$$\sum_j w_j\Bigg(- y_j - \overline{y_j} + 2a_{0R} + 2a_{1R}\frac{1}{k_j} + 2a_{2R}k_j + 2a_{3R}k_j^2 + a_4\bigg(\frac{y_j + \overline{y_j}}{k_j}\bigg)\Bigg) = 0$$

$$\sum_j w_j\Bigg(- y_{Rj} + a_{0R} + a_{1R}\frac{1}{k_j} + a_{2R}k_j + a_{3R}k_j^2 + a_4\frac{y_{Rj}}{k_j}\Bigg) = 0$$

where the notation $y_{Rj} \equiv Real(y_j)$ and $y_{Ij} \equiv Imaginary(y_j)$.

$$\frac{\partial}{\partial a_{0I}}\sum_j w_j(...)(...) = 0 =$$
$$\sum_j w_j\Big(i\, y_j - i\, a_{0R} + a_{0I} - i\, a_{1R}\tfrac{1}{k_j} + a_{1I}\tfrac{1}{k_j} - i\, a_{2R}k_j + a_{2I}k_j - i\, a_{3R}k_j^2 + a_{3I}k_j^2 - i\, a_4\tfrac{y_j}{k_j}\Big)$$
$$+ w_j\Big(- i\,\overline{y_j} + i\, a_{0R} + a_{0I} + i\, a_{1R}\tfrac{1}{k_j} + a_{1I}\tfrac{1}{k_j} + i\, a_{2R}k_j + a_{2I}k_j + i\, a_{3R}k_j^2 + a_{3I}k_j^2 + i\, a_4\tfrac{\overline{y_j}}{k_j}\Big)$$
$$\sum_j w_j\Bigg(i\,(y_j - \overline{y_j}) + 2a_{0I} + 2a_{1I}\frac{1}{k_j} + 2a_{2I}k_j + 2a_{3I}k_j^2 - i\, a_4\frac{(y_j - \overline{y_j})}{k_j}\Bigg) = 0$$

$$\sum_j w_j\Bigg(- y_{Ij} + a_{0I} + a_{1I}\frac{1}{k_j} + a_{2I}k_j + a_{3I}k_j^2 + a_4\frac{y_{Ij}}{k_j}\Bigg) = 0$$

$$\frac{\partial}{\partial a_{1R}}\sum_j w_j(...)(...) = 0 =$$
$$\sum_j w_j\Big(- \tfrac{y_j}{k_j} + a_{0R}\tfrac{1}{k_j} + i\, a_{0I}\tfrac{1}{k_j} + a_{1R}\tfrac{1}{k_j^2} + i\, a_{1I}\tfrac{1}{k_j^2} + a_{2R} + i\, a_{2I} + a_{3R}k_j + i\, a_{3I}k_j + a_4\tfrac{y_j}{k_j^2}\Big)$$
$$+ \Big(- \tfrac{\overline{y_j}}{k_j} + a_{0R}\tfrac{1}{k_j} - i\, a_{0I}\tfrac{1}{k_j} + a_{1R}\tfrac{1}{k_j^2} - i\, a_{1I}\tfrac{1}{k_j^2} + a_{2R} - i\, a_{2I} + a_{3R}k_j - i\, a_{3I}k_j + a_4\tfrac{\overline{y_j}}{k_j^2}\Big)$$

$$\sum_j w_j \left(-\left(\frac{y_j + \overline{y_j}}{k_j}\right) + 2a_{0R}\frac{1}{k_j} + 2a_{1R}\frac{1}{k_j^2} + 2a_{2R} + 2a_{3R}k_j + a_4\left(\frac{y_j + \overline{y_j}}{k_j^2}\right)\right) = 0$$

$$\sum_j w_j \left(-\frac{y_{Rj}}{k_j} + a_{0R}\frac{1}{k_j} + a_{1R}\frac{1}{k_j^2} + a_{2R} + a_{3R}k_j + a_4\frac{y_{Rj}}{k_j^2}\right) = 0$$

$$\frac{\partial}{\partial a_{1I}}\sum_j w_j(...)(...) = 0 =$$
$$\sum_j w_j\Big(i\tfrac{y_j}{k_j} - i\,a_{0R}\tfrac{1}{k_j} + \ a_{0I}\tfrac{1}{k_j} - i\,a_{1R}\tfrac{1}{k_j^2} + \ a_{1I}\tfrac{1}{k_j^2} - i\,a_{2R} + a_{2I} - i\,a_{3R}k_j + a_{3I}k_j - i\,a_4\tfrac{y_j}{k_j^2}\Big)$$

$$+\Big(-i\tfrac{\overline{y_j}}{k_j} + i\,a_{0R}\tfrac{1}{k_j} + a_{0I}\tfrac{1}{k_j} + i\,a_{1R}\tfrac{1}{k_j^2} + a_{1I}\tfrac{1}{k_j^2} + i\,a_{2R} + \ a_{2I} + i\,a_{3R}k_j + \ a_{3I}k_j + i\,a_4\tfrac{\overline{y_j}}{k_j^2}\Big)$$

$$\sum_j w_j\left(i\left(\frac{y_j - \overline{y_j}}{k_j}\right) + 2\,a_{0I}\frac{1}{k_j} + 2\,a_{1I}\frac{1}{k_j^2} + 2a_{2I} + 2a_{3I}k_j - i\,a_4\left(\frac{y_j - \overline{y_j}}{k_j^2}\right)\right) = 0$$

$$\sum_j w_j\left(-\frac{y_{Ij}}{k_j} + \ a_{0I}\frac{1}{k_j} + \ a_{1I}\frac{1}{k_j^2} + a_{2I} + a_{3I}k_j + a_4\frac{y_{Ij}}{k_j^2}\right) = 0$$

$$\frac{\partial}{\partial a_{2R}}\sum_j w_j(...)(...) = 0 =$$
$$\sum_j w_j\big(-y_jk_j + a_{0R}k_j + i\,a_{0I}k_j + a_{1R} + i\,a_{1I} + a_{2R}k_j^2 + i\,a_{2I}{k_j}^2 + a_{3R}k_j^3 + i\,a_{3I}k_j^3 + a_4y_j\big)$$
$$+\big(-\overline{y_j}k_j + a_{0R}k_j - i\,a_{0I}k_j + a_{1R} - i\,a_{1I} + a_{2R}k_j^2 - i\,a_{2I}k_j^2 + a_{3R}k_j^3 - i\,a_{3I}k_j^3 + a_4\overline{y_j}\big)$$

$$\sum_j w_j\big(-(y_j + \overline{y_j})k_j + 2a_{0R}k_j + 2a_{1R} + 2a_{2R}k_j^2 + 2a_{3R}k_j^3 + a_4(y_j + \overline{y_j})\big) = 0$$

$$\sum_j w_j\big(-y_{Rj}k_j + a_{0R}k_j + a_{1R} + a_{2R}k_j^2 + a_{3R}k_j^3 + a_4y_{Rj}\big) = 0$$

$$\frac{\partial}{\partial a_{2I}}\sum_j w_j(...)(...) = 0 =$$
$$\sum_j w_j\big(i\,y_jk_j - i\,a_{0R}k_j + a_{0I}k_j - i\,a_{1R} + a_{1I} - i\,a_{2R}k_j^2 + a_{2I}k_j^2 - i\,a_{3R}k_j^3 + a_{3I}k_j^3 - i\,a_4y_j\big)$$
$$+\big(-i\,\overline{y_j}k_j + i\,a_{0R}k_j + a_{0I}k_j + i\,a_{1R} + a_{1I} + i\,a_{2R}k_j^2 + a_{2I}k_j^2 + i\,a_{3R}k_j^3 + \ a_{3I}k_j^3 + i\,a_4\overline{y_j}\big)$$

$$\sum_j w_j\big(i\,(y_j-\overline{y_j})k_j+2a_{0I}k_j+2a_{1I}+2a_{2I}k_j^2+2a_{3I}k_j^3-i\,a_4(y_j-\overline{y_j})\big)=0$$

$$\sum_j w_j\big(-\ y_{Ij}k_j+a_{0I}k_j+a_{1I}+a_{2I}k_j^2+a_{3I}k_j^3+\ a_4y_{Ij}\big)=0$$

$$\frac{\partial}{\partial a_{3R}}\sum_j w_j(\ldots)(\ldots)=0=$$
$$\sum_j w_j\big(-y_jk_j^2+a_{0R}k_j^2+i\,a_{0I}k_j^2+a_{1R}k_j+i\,a_{1I}k_j+a_{2R}k_j^3+i\,a_{2I}k_j^3+a_{3R}k_j^4+i\,a_{3I}k_j^4+a_4y_jk_j\big)$$
$$+\big(-\overline{y_j}k_j^2+a_{0R}k_j^2-i\,a_{0I}k_j^2+a_{1R}k_j-i\,a_{1I}k_j+a_{2R}k_j^3-i\,a_{2I}k_j3+a_{3R}k_j^4-i\,a_{3I}k_j^4+a_4\overline{y_j}k_j\big)$$

$$\sum_j w_j\big(-(y_j+\overline{y_j})k_j^2+2a_{0R}k_j^2+2a_{1R}k_j+2a_{2R}k_j^3+2a_{3R}k_j^4+a_4(y_j+\overline{y_j})k_j\big)=0$$

$$\sum_j w_j\big(-y_{Rj}k_j^2+a_{0R}k_j^2+a_{1R}k_j+a_{2R}k_j^3+a_{3R}k_j^4+a_4y_{Rj}k_j\big)=0$$

$$\frac{\partial}{\partial a_{3I}}\sum_j w_j(\ldots)(\ldots)=0=$$
$$\sum_j w_j\big(i\,y_jk_j^2-i\,a_{0R}k_j^2+\ a_{0I}k_j^2-i\,a_{1R}k_j+\ a_{1I}k_j-i\,a_{2R}k_j^3+\ a_{2I}k_j^3-i\,a_{3R}k_j^4+\ a_{3I}k_j^4-i\,a_4y_jk_j\big)$$
$$+\big(-i\,\overline{y_j}k_j^2+i\,a_{0R}k_j^2+\ a_{0I}k_j^2+i\,a_{1R}k_j+\ a_{1I}k_j+i\,a_{2R}k_j^3+\ a_{2I}k_j^3+i\,a_{3R}k_j^4+a_{3I}k_j^4+i\,a_4\overline{y_j}k_j\big)$$

$$\sum_j w_j\big(i\,(y_j-\overline{y_j})k_j^2+\ 2a_{0I}k_j^2+\ 2a_{1I}k_j+\ 2a_{2I}k_j^3+2\,a_{3I}k_j^4-i\,a_4(y_j-\overline{y_j})k_j\big)=0$$

$$\sum_j w_j\big(-y_{Ij}\,k_j^2+a_{0I}k_j^2+a_{1I}k_j+a_{2I}k_j^3+a_{3I}k_j^4+\ a_4y_{Ij}\,k_j\big)=0$$

$$\frac{\partial}{\partial a_{3I}}\sum_j w_j(\ldots)(\ldots)=0=\ -\frac{\overline{y_j}}{k_j}\quad-\frac{y_j}{k_j}$$
$$\sum_j w_j\Big(-\frac{y_j\overline{y_j}}{k_j}+a_{0R}\frac{\overline{y_j}}{k_j}+i\,a_{0I}\frac{\overline{y_j}}{k_j}+a_{1R}\frac{\overline{y_j}}{k_j^2}+i\,a_{1I}\frac{\overline{y_j}}{k_j^2}+a_{2R}\overline{y_j}+i\,a_{2I}\overline{y_j}+a_{3R}\overline{y_j}k_j+i\,a_{3I}\overline{y_j}k_j+a_4\frac{y_j\overline{y_j}}{k_j^2}\Big)$$
$$+\Big(-\frac{y_j\overline{y_j}}{k_j}+a_{0R}\frac{y_j}{k_j}-i\,a_{0I}\frac{y_j}{k_j}+a_{1R}\frac{y_j}{k_j^2}-i\,a_{1I}\frac{y_j}{k_j^2}+a_{2R}y_j-i\,a_{2I}y_j+a_{3R}y_jk_j-i\,a_{3I}y_jk_j+a_4\frac{y_j\overline{y_j}}{k_j^2}\Big)$$

$$\sum_j w_j\Big(-2\frac{y_j\overline{y_j}}{k_j}+a_{0R}\frac{\overline{y_j}+y_j}{k_j}+i\,a_{0I}\frac{\overline{y_j}-y_j}{k_j}+a_{1R}\frac{\overline{y_j}+y_j}{k_j^2}+i\,a_{1I}\frac{\overline{y_j}-y_j}{k_j^2}+a_{2R}(\overline{y_j}+y_j)$$

$$+i\,a_{2I}(\overline{y_j}-y_j)+a_{3R}(\overline{y_j}+y_j)k_j+i\,a_{3I}(\overline{y_j}-y_j)k_j+2a_4\frac{y_j\overline{y_j}}{k_j^2}\Big)=0$$

$$\sum_j w_j\Big(-\frac{y_j\overline{y_j}}{k_j}+a_{0R}\frac{y_{Rj}}{k_j}+\ a_{0I}\frac{y_{Ij}}{k_j}+a_{1R}\frac{y_{Rj}}{k_j^2}+a_{1I}\frac{y_{Ij}}{k_j^2}+a_{2R}y_{Rj}+a_{2I}y_{Ij}+$$

$$a_{3R}y_{Rj}k_j+a_{3I}y_{Ij}k_j+a_4\frac{y_j\overline{y_j}}{k_j^2}\Big)=0$$

So that the collected nine equations in nine unknowns are:

$$\sum_j w_j\left(-y_{Rj}+a_{0R}+a_{1R}\frac{1}{k_j}+a_{2R}k_j+a_{3R}k_j^2+a_4\frac{y_{Rj}}{k_j}\right)=0$$

$$\sum_j w_j\left(-y_{Ij}\ +\ a_{0I}+\ a_{1I}\frac{1}{k_j}+\ a_{2I}k_j+\ a_{3I}k_j^2+a_4\frac{y_{Ij}}{k_j}\right)=0$$

$$\sum_j w_j\left(-\frac{y_{Rj}}{k_j}+a_{0R}\frac{1}{k_j}+a_{1R}\frac{1}{k_j^2}+a_{2R}+a_{3R}k_j+a_4\frac{y_{Rj}}{k_j^2}\right)=0$$

$$\sum_j w_j\left(-\frac{y_{Ij}}{k_j}+\ a_{0I}\frac{1}{k_j}+\ a_{1I}\frac{1}{k_j^2}+a_{2I}+a_{3I}k_j+a_4\frac{y_{Ij}}{k_j^2}\right)=0$$

$$\sum_j w_j\big(-y_{Rj}k_j+a_{0R}k_j+a_{1R}+a_{2R}k_j^2+a_{3R}k_j^3+a_4y_{Rj}\big)=0$$

$$\sum_j w_j\big(-\ y_{Ij}k_j+a_{0I}k_j+a_{1I}+a_{2I}k_j^2+a_{3I}k_j^3+\ a_4y_{Ij}\big)=0$$

$$\sum_j w_j\big(-y_{Rj}k_j^2+a_{0R}k_j^2+a_{1R}k_j+a_{2R}k_j^3+a_{3R}k_j^4+a_4y_{Rj}k_j\big)=0$$

$$\sum_j w_j\big(-y_{Ij}\,k_j^2+a_{0I}k_j^2+a_{1I}k_j+a_{2I}k_j^3+a_{3I}k_j^4+\ a_4y_{Ij}\,k_j\big)=0$$

$$\sum_j w_j(-\frac{y_j\overline{y_j}}{k_j} + a_{0R}\frac{y_{Rj}}{k_j} + \; a_{0I}\frac{y_{Ij}}{k_j} + a_{1R}\frac{y_{Rj}}{k_j^2} + a_{1I}\frac{y_{Ij}}{k_j^2} + a_{2R}y_{Rj} + a_{2I}y_{Ij} +$$

$$a_{3R}y_{Rj}k_j + a_{3I}y_{Ij}k_j + a_4\frac{y_j\overline{y_j}}{k_j^2}) = 0$$

In matrix form, where each element represents a sum over j:

$$\begin{bmatrix} w_j & 0 & w_j\frac{1}{k_j} & 0 & w_jk_j & 0 & w_jk_j^2 & 0 & w_j\frac{y_{Rj}}{k_j} \\ 0 & w_j & 0 & w_j\frac{1}{k_j} & 0 & w_jk_j & 0 & w_jk_j^2 & w_j\frac{y_{Ij}}{k_j} \\ w_j\frac{1}{k_j} & 0 & w_j\frac{1}{k_j^2} & 0 & w_j & 0 & w_jk_j & 0 & w_j\frac{y_{Rj}}{k_j^2} \\ 0 & w_j\frac{1}{k_j} & 0 & w_j\frac{1}{k_j^2} & 0 & w_j & 0 & w_jk_j & w_j\frac{y_{Ij}}{k_j^2} \\ w_jk_j & 0 & w_j & 0 & w_jk_j^2 & 0 & w_jk_j^3 & 0 & w_jy_{Rj} \\ 0 & w_jk_j & 0 & w_j & 0 & w_jk_j^2 & 0 & w_jk_j^3 & w_jy_{Ij} \\ w_jk_j^2 & 0 & w_jk_j & 0 & w_jk_j^3 & 0 & w_jk_j^4 & 0 & w_jy_{Rj}k_j \\ 0 & w_jk_j^2 & 0 & w_jk_j & 0 & w_jk_j^3 & 0 & w_jk_j^4 & w_jy_{Ij}\,k_j \\ w_j\frac{y_{Rj}}{k_j} & w_j\frac{y_{Ij}}{k_j} & w_j\frac{y_{Rj}}{k_j^2} & w_j\frac{y_{Ij}}{k_j^2} & w_jy_{Rj} & w_jy_{Ij} & w_jy_{Rj}k_j & w_jy_{Ij}k_j & w_j\frac{|y_j|^2}{k_j^2} \end{bmatrix} \begin{bmatrix} a_{0R} \\ a_{0I} \\ a_{1R} \\ a_{1I} \\ a_{2R} \\ a_{2I} \\ a_{3R} \\ a_{3I} \\ a_4 \end{bmatrix} = \begin{bmatrix} w_jy_{Rj} \\ w_jy_{Ij} \\ w_j\frac{y_{Rj}}{k_j} \\ w_j\frac{y_{Ij}}{k_j} \\ w_jy_{Rj}k_j \\ w_jy_{Ij}k_j \\ w_jy_{Rj}k_j^2 \\ w_jy_{Ij}\,k_j^2 \\ w_j\frac{|y_j|^2}{k_j} \end{bmatrix} \tag{13}$$

This system is too complicated to satisfactorily solve symbolically but can be solved easily using numeric matrix inversion or another linear system solution method on a case-by-case basis. As before, the weighting factor is available to adjust the method for better performance. This method has been given preliminary tests in which it appears to perform quite well in determining precise frequencies and amplitudes for mixed signals.

F in	F out	Amplitude in		Amplitude out	
22.585	22.585	346.230	-234.323	346.230	-234.322
71.479	71.479	-176.693	21.150	-176.692	21.152
206.245	206.245	314.081	294.419	314.081	294.419
774.578	774.578	-141.360	390.191	-141.360	390.191
1448.484	1447.909	-360.471	-342.341	2241.656	-920.017
1616.000	1616.000	385.323	-145.001	385.323	-145.001
1758.175	1758.175	-187.616	-441.476	-187.616	-441.476
2349.990	2349.990	314.202	-399.130	314.202	-399.130
2758.343	2758.343	188.560	-348.127	188.560	-348.127
2905.176	2905.176	168.015	-348.763	168.015	-348.763
3109.001	3108.998	183.427	147.184	178.753	152.784
3119.557	3119.557	178.925	140.215	178.924	140.216
3296.036	3296.036	69.103	-116.692	69.103	-116.692

Table II
Sample Results of Platform Least Squares Fit

Table II shows that the results of this platform method are, in a word, amazing. At this point, these are typical results in the same sense that the results shown in Table I are typical for that method. It will be noticed in the above that one component, near $f = 1448$, is glaringly in error. At this point, this happens. Here it is due to two random frequencies being too close together. The second frequency is completely unreported. Sources of such errors and methods for dealing with them are being pursued and it is fully expected that satisfactory resolutions will be found.

The above methods deal with single frequencies, but in some cases it is possible to deal with multiple frequencies still using linear least squares. For example, with the frequencies known, a linear least squares to the corresponding amplitudes is quite feasible:

$$y = A_0' + \frac{A_1'}{k+\epsilon_1} + \frac{A_2'}{k+\epsilon_2} + ... + \frac{A_n'}{k+\epsilon_n} \tag{14}$$

where the ϵ_i are known is in simple linear least squares form. It is also possible to determine a second frequency given known nearby frequencies.

$$y = \frac{A'}{(k+\epsilon)} + A_0' + \frac{A_1'}{k+\epsilon_1} + \frac{A_2'}{k+\epsilon_2} \tag{15}$$

where ϵ_1 and ϵ_2 and perhaps more frequencies are known and included. Equation (15) can be treated in the same manner as Equation (3) was treated above. In fact it would be quite feasible to add further platform correction terms as was done in Equation (3) if this proves desirable. Furthermore, it is also quite feasible to perform a nonlinear least squares fit using one of the above methods to obtain most of the coefficients using linear least squares and a nonlinear method to vary the "known" frequencies in order to determine an overall minimum sum of squares.

Multiple Adjacent Frequencies

Here the method is extended to the general case of multiple closely spaced frequencies

The form of Equation 13 from the first part above,

$$y = \frac{A'}{(k+\epsilon)} \qquad [13](1)$$

was extended to the form

$$y = \frac{A'}{(k+\epsilon)} + C + Sk + Tk^2 \qquad [3](2)$$

of Equation 3 in the second part (Platform Least Squares) above.

In Platform Least Squares, it was remarked that the situation of two (or more) nearby frequencies could cause errors in an analysis that otherwise performed remarkably well. In general, this problem becomes important if the there are two or more precise frequency signals of significant amplitude within the range of frequencies represented by the span of coefficients from the FFT or DFT that are used as data for the least squares fit. The root assumption of this method is "at any instant, the signal which we wish to analyze is comprised of a small number of specific frequency components," which we believe to be the case for many everyday signals. Nonetheless, there is nothing to preclude several of this small number from being adjacent to one another. For many practical purposes it will be sufficient to report such groupings as a single frequency. The ear does not separately detect the three frequencies of a note in the upper octaves of a piano, but will detect the "beat" as a slow variation of amplitude of the single frequency, which this method is also capable of tracking. Nonetheless, it is likely that there will be cases in which it is more desirable to separate groupings of a few nearby frequencies.

The most obvious way of dealing with the situation of several nearby frequencies is to choose the span of data so that it avoids multiple frequencies, even if this requires that the sample be less symmetric about the target frequency. However, it is also possible to fit multiple precise frequencies that occur within a span, as will be shown. These frequencies can even fall in a single gap between two adjacent frequencies of the DFT, although it is to be expected that as the signals become more similar, separating them will become less accurate. Each new frequency that is added adds a complex amplitude coefficient, A'_p, and a real frequency, ϵ_p, which means that to maintain the same excess degrees of freedom in the least squares fit, $1\frac{1}{2}$ new points must be attached to the span, widening it and potentially including more frequencies, which fact must be balanced against the ability to detect an additional frequency. The $1\frac{1}{2}$ point requirement comes about because each frequency adds three new parameters; two from the complex amplitude and one from the real frequency, while each additional point from the DFT span adds only two values, from the single complex Fourier coefficient.

We address the case of nearby frequencies using the form

$$y = \sum_p \frac{A'_p}{(k+\epsilon_p)} + C + Sk + Tk^2 \tag{3}$$

for the case of a cluster of p nearby frequencies. First we will deal with the case of just two nearby frequencies and then with three in order to develop a general formula.

$$y = \frac{A'_1}{(k+\epsilon_1)} + \frac{A'_2}{(k+\epsilon_2)} + C + Sk + Tk^2 \tag{4}$$

Multiplying out, we get

$$yk^2 + yk(\epsilon_1+\epsilon_2) + \epsilon_1\epsilon_2 y = A_1{}'\epsilon_2 + A_2{}'\epsilon_1 + (A_1{}' + A_2{}')k + Ck^2 + Ck(\epsilon_1+\epsilon_2) + C\epsilon_1\epsilon_2 +$$

$$Sk^3 + Sk^2(\epsilon_1+\epsilon_2) + Sk\epsilon_1\epsilon_2 + Tk^4 + Tk^3(\epsilon_1+\epsilon_2) + Tk^2\epsilon_1\epsilon_2 \tag{5}$$

dividing by k^2

$$y = [C + S(\epsilon_1+\epsilon_2) + T\epsilon_1\epsilon_2] + (A_1{}'\epsilon_2 + A_2{}'\epsilon_1 + C\epsilon_1\epsilon_2)\frac{1}{k^2} +$$

$$[C(\epsilon_1+\epsilon_2) + S\epsilon_1\epsilon_2 + (A_1{}' + A_2{}')]\frac{1}{k} + [T(\epsilon_1+\epsilon_2) + S]k + Tk^2 - (\epsilon_1+\epsilon_2)\frac{y}{k} - \epsilon_1\epsilon_2\frac{y}{k^2} \tag{6}$$

This is in the linear form

$$y = a_0 + a_1\frac{1}{k^2} + a_2\frac{1}{k} + a_3 k + a_4 k^2 + a_5\frac{y}{k} + a_6\frac{y}{k^2} \tag{7}$$

where $a_0 ... a_4$ are complex and a_5 and a_6, which correspond to $-(\epsilon_1+\epsilon_2)$ and $-\epsilon_1\epsilon_2$, are real only. When the constants $a_0 ... a_6$ are obtained, then it can be recognized that $(x-\epsilon_1)(x-\epsilon_2) = x^2 - (\epsilon_1+\epsilon_2)x + \epsilon_1\epsilon_2$ and thus ϵ_1 and ϵ_2 are the roots of the standard quadratic $x^2 + a_5 x - a_6 = 0$.

$$\epsilon_1 = \frac{-a_5 - \sqrt{a_5^2 + 4a_6}}{2} \tag{8}$$

$$\epsilon_2 = \frac{-a_5 + \sqrt{a_5^2 + 4a_6}}{2} \tag{9}$$

$$T = a_4 \tag{10}$$

$$S = a_3 - T(\epsilon_1+\epsilon_2) \tag{11}$$

$$C = a_0 - S(\epsilon_1 + \epsilon_2) - T\epsilon_1\epsilon_2 \tag{12}$$

$$C(\epsilon_1 + \epsilon_2) + S\epsilon_1\epsilon_2 + ({A_1}' + {A_2}') = a_2$$

$${A_2}' = a_2 - C(\epsilon_1 + \epsilon_2) - S\epsilon_1\epsilon_2 - {A_1}'$$

$${A_1}'\epsilon_2 + {A_2}'\epsilon_1 + C\epsilon_1\epsilon_2 = a_1$$

$${A_1}'\epsilon_2 = a_1 - {A_2}'\epsilon_1 - C\epsilon_1\epsilon_2$$

$${A_1}'\epsilon_2 = a_1 - a_2\epsilon_1 + C(\epsilon_1 + \epsilon_2)\epsilon_1 + S\epsilon_1^2\epsilon_2 + {A_1}'\epsilon_1 - C\epsilon_1\epsilon_2$$

$${A_1}'(\epsilon_2 - \epsilon_1) = a_1 - a_2\epsilon_1 + C\epsilon_1^2 + S\epsilon_1^2\epsilon_2$$

$${A_1}' = \frac{C\epsilon_1^2 + S\epsilon_1^2\epsilon_2 + a_1 - a_2\epsilon_1}{\epsilon_2 - \epsilon_1} \tag{13}$$

$${A_2}' = \frac{C\epsilon_2^2 + S\epsilon_2^2\epsilon_1 + a_1 - a_2\epsilon_2}{\epsilon_1 - \epsilon_2} \tag{14}$$

as before the original

$$A_p = -2\pi i A_p' / \left(1 - e^{i2\pi\epsilon_p}\right)$$

can be obtained from the result.

Now we con continue on to the case of three nearby frequencies in order to see how the system behaves as we add frequencies.

$$y = \frac{A_1'}{(k + \epsilon_1)} + \frac{A_2'}{(k + \epsilon_2)} + \frac{A_3'}{(k + \epsilon_3)} + C + Sk + Tk^2 \tag{15}$$

Multiplying out, we get

$$yk^3 + yk^2(\epsilon_1 + \epsilon_2 + \epsilon_3) + yk(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) + y\epsilon_1\epsilon_2\epsilon_3 =$$

$$A_1'k^2 + A_1'k(\epsilon_2 + \epsilon_3) + A_1'\epsilon_2\epsilon_3 + A_2'k^2 + A_2'k(\epsilon_1 + \epsilon_3) + A_2'\epsilon_1\epsilon_3 +$$

$$A_3'k^2 + A_3'k(\epsilon_1 + \epsilon_2) + A_3'\epsilon_1\epsilon_2$$

$$Ck^3 + Ck^2(\epsilon_1 + \epsilon_2 + \epsilon_3) + Ck(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) + C\epsilon_1\epsilon_2\epsilon_3 +$$

$$Sk^4 + Sk^3(\epsilon_1+\epsilon_2+\epsilon_3) + Sk^2(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3) + Sk\epsilon_1\epsilon_2\epsilon_3 +$$

$$Tk^5 + Tk^4(\epsilon_1+\epsilon_2+\epsilon_3) + Tk^3(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3) + Tk^2\epsilon_1\epsilon_2\epsilon_3 \quad (16)$$

dividing by k^3

$$y = [C + S(\epsilon_1+\epsilon_2+\epsilon_3) + T(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3)] +$$

$$(A_1'\epsilon_2\epsilon_3 + A_2'\epsilon_1\epsilon_3 + A_3'\epsilon_1\epsilon_2 + C\epsilon_1\epsilon_2\epsilon_3)\frac{1}{k^3} +$$

$$+ [A_1'(\epsilon_2+\epsilon_3) + A_2'(\epsilon_1+\epsilon_3) + A_3'(\epsilon_1+\epsilon_2) + C(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3) + S\epsilon_1\epsilon_2\epsilon_3]\frac{1}{k^2}$$

$$+ [(A_1' + A_2' + A_3') + C(\epsilon_1+\epsilon_2+\epsilon_3) + S(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3) + T\epsilon_1\epsilon_2\epsilon_3]\frac{1}{k}$$

$$+ [S + T(\epsilon_1+\epsilon_2+\epsilon_3)]k + Tk^2$$

$$- (\epsilon_1+\epsilon_2+\epsilon_3)\frac{y}{k} - (\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3)\frac{y}{k^2} - \epsilon_1\epsilon_2\epsilon_3\frac{y}{k^3} \quad (17)$$

This is in the linear form

$$y = a_0 + a_1\frac{1}{k^3} + a_2\frac{1}{k^2} + a_3\frac{1}{k} + a_4k + a_5k^2 + a_6\frac{y}{k} + a_7\frac{y}{k^2} + a_8\frac{y}{k^3} \quad (18)$$

where $a_0 ... a_5$ are complex and $a_6 ... a_8$, which correspond to $-(\epsilon_1+\epsilon_2+\epsilon_3)$, $-(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3)$ and $-\epsilon_1\epsilon_2\epsilon_3$, are real only. When the constants $a_0 ... a_8$ are obtained, then

$$\epsilon_1\epsilon_2\epsilon_3 = -a_8$$

$$\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3 = -a_7$$

$$\epsilon_1+\epsilon_2+\epsilon_3 = -a_6$$

These can be recognized as the standard form coefficients for a cubic equation with the roots ϵ_1, ϵ_2, and ϵ_3.

$$x^3 + a_6x^2 - a_7x + a_8 = (x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3) = 0 \quad (19)$$

This equation can be solved directly using the cubic solution (Cardan's solution)

$$z = -36a_6a_7 - 108a_8 - 8a_6^3 + 12\sqrt{-12a_7^3 - 3a_7^2a_6^2 + 54a_6a_7a_8 + 81a_8^2 + 12a_8a_6^3}$$

$$w = \frac{-\frac{a_7}{3} - \frac{a_6^2}{9}}{z^{\frac{1}{3}}}$$

$$\epsilon_1 = \frac{z^{\frac{1}{3}}}{6} - 6w - \frac{a_6}{3}$$

$$\epsilon_2 = -\frac{z^{\frac{1}{3}}}{12} + 3w - \frac{a_6}{3} - \frac{i\sqrt{3}}{2}\left(\frac{z^{\frac{1}{3}}}{6} + 6w\right)$$

$$\epsilon_3 = -\frac{z^{\frac{1}{3}}}{12} + 3w - \frac{a_6}{3} + \frac{i\sqrt{3}}{2}\left(\frac{z^{\frac{1}{3}}}{6} + 6w\right)$$

For this to be valid all three ϵ_p must be real.

$$T = a_5 \tag{20}$$

$$S = a_4 - T(\epsilon_1 + \epsilon_2 + \epsilon_3) \tag{21}$$

$$C = a_0 - S(\epsilon_1 + \epsilon_2 + \epsilon_3) - T(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) \tag{22}$$

$$A_1'\epsilon_2\epsilon_3 + A_2'\epsilon_1\epsilon_3 + A_3'\epsilon_1\epsilon_2 + C\epsilon_1\epsilon_2\epsilon_3 = a_1$$

$$A_1'(\epsilon_2 + \epsilon_3) + A_2'(\epsilon_1 + \epsilon_3) + A_3'(\epsilon_1 + \epsilon_2) + C(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) + S\epsilon_1\epsilon_2\epsilon_3 = a_2$$

$$(A_1' + A_2' + A_2') + C(\epsilon_1 + \epsilon_2 + \epsilon_3) + S(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) + T\epsilon_1\epsilon_2\epsilon_3 = a_3$$

$$A_1' = \frac{[S(\epsilon_2 + \epsilon_3) + T\epsilon_2\epsilon_3 + C]\epsilon_1^3 - a_1 + a_2\epsilon_1 - a_3\epsilon_1^2}{(\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_1)} \tag{23}$$

$$A_2' = \frac{[S(\epsilon_1 + \epsilon_3) + T\epsilon_1\epsilon_3 + C]\epsilon_2^3 - a_1 + a_2\epsilon_2 - a_3\epsilon_2^2}{(\epsilon_2 - \epsilon_3)(\epsilon_1 - \epsilon_2)} \tag{24}$$

$$A_3' = \frac{[S(\epsilon_1 + \epsilon_2) + T\epsilon_1\epsilon_2 + C]\epsilon_3^3 - a_1 + a_2\epsilon_3 - a_3\epsilon_3^2}{(\epsilon_3 - \epsilon_1)(\epsilon_2 - \epsilon_3)} \tag{25}$$

As in the case with two frequencies, once A_1' is obtained, A_2' and A_3' can be obtained through symmetry, and as before the original

$$A_p = -2\pi i A_p' / \left(1 - e^{i2\pi\epsilon_p}\right) \tag{26}$$

can be obtained from the result.

Clearly, more frequencies can be added using a similar technique. The fitting equation will remain linear with respect to constants and in the same form and each added frequency will require the solution of a standard polynomial of one higher order for the frequencies and a set of linear equations of one higher order for the complex amplitudes. This solution need not be done algebraically; it can be done numerically if that is easier.

Looking at Equation [7] from Supplement 1 and Equations (7) and (18) from this report

$$y = a_0 + a_1\frac{1}{k} + a_2 k + a_3 k^2 + a_4\frac{y}{k} \qquad [7]$$

$$y = a_0 + a_1\frac{1}{k^2} + a_2\frac{1}{k} + a_3 k + a_4 k^2 + a_5\frac{y}{k} + a_6\frac{y}{k^2} \qquad (7)$$

$$y = a_0 + a_1\frac{1}{k^3} + a_2\frac{1}{k^2} + a_3\frac{1}{k} + a_4 k + a_5 k^2 + a_6\frac{y}{k} + a_7\frac{y}{k^2} + a_8\frac{y}{k^3} \qquad (18)$$

these can be expressed as an equation of common form:

$$y = a_0 + a_1 k + a_2 k^2 + \sum_{p=1}^{n}\left[a_{p+3}\frac{1}{k^p} + b_p\frac{y}{k^p}\right] \qquad (27)$$

where $p = 1, 2, 3$ for the above cases and is extended for cases of more frequencies. The coefficients a are complex and b are real.

This corresponds to

$$y = C + Sk + Tk^2 + \sum_{p=1}^{n}\frac{A'_p}{(k+\epsilon_p)} \qquad (28)$$

and in general the ϵ_p are the solutions of

$$x^n + \sum_{p=1}^{n}(-1)^{p+1} b_{n-p} x^{n-p} = 0 \qquad (29)$$

which we have shown for the linear, quadratic, and cubic case to be:

$$\epsilon_1 = -b_1 \qquad (30)$$

$$\epsilon_1 = \frac{-b_1 - \sqrt{b_1^2 + 4b_2}}{2} \qquad (31)$$

$$\epsilon_2 = \frac{-b1 + \sqrt{b_1^2 + 4b_2}}{2} \tag{32}$$

$$z = -36b_1b_2 - 108b_3 - 8b_1^3 + 12\sqrt{-12b_2^3 - 3b_2^2b_1^2 + 54b_1b_2b_3 + 81b_3^2 + 12b_3b_1^3}$$

$$w = \frac{-\frac{b_2}{3} - \frac{b_1^2}{9}}{z^{\frac{1}{3}}}$$

$$\epsilon_1 = \frac{z^{\frac{1}{3}}}{6} - 6w - \frac{b_1}{3} \tag{33}$$

$$\epsilon_2 = -\frac{z^{\frac{1}{3}}}{12} + 3w - \frac{b_1}{3} - \frac{i\sqrt{3}}{2}\left(\frac{z^{\frac{1}{3}}}{6} + 6w\right) \tag{34}$$

$$\epsilon_3 = -\frac{z^{\frac{1}{3}}}{12} + 3w - \frac{b_1}{3} + \frac{i\sqrt{3}}{2}\left(\frac{z^{\frac{1}{3}}}{6} + 6w\right) \tag{35}$$

and from combining other results:

$$T = a_2 \tag{36}$$

$$S = a_1 - T\sum_{p=1}^{n}\epsilon_p \tag{37}$$

$$C = a_0 - S\sum_{p=1}^{n}\epsilon_p - T\sum_{q=1}^{n}\sum_{r=q+1}^{n}\epsilon_r\epsilon_s \tag{38}$$

The coefficients for $a_3 ... a_{n+2}$ along with the above results will formulate into a series of n linear equations in the n unknowns, A'_{p}. Corresponding to the linear, quadratic, and cubic cases for ϵ above, the solutions are:

$$A'_1 = a_3 - C\epsilon_1 \tag{12}$$

$$A'_1 = \frac{C\epsilon_1^2 + S\epsilon_1^2\epsilon_2 + a_4 - a_3\epsilon_1}{\epsilon_2 - \epsilon_1} \tag{13}$$

$$A'_2 = \frac{C\epsilon_2^2 + S\epsilon_2^2\epsilon_1 + a_4 - a_3\epsilon_2}{\epsilon_1 - \epsilon_2} \tag{14}$$

$$A_1' = \frac{[S(\epsilon_2 + \epsilon_3) + T\epsilon_2\epsilon_3 + C]\epsilon_1^3 - a_5 + a_4\epsilon_1 - a_3\epsilon_1^2}{(\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_1)} \tag{23}$$

$$A_2' = \frac{[S(\epsilon_1 + \epsilon_3) + T\epsilon_1\epsilon_3 + C]\epsilon_2^3 - a_5 + a_4\epsilon_2 - a_3\epsilon_2^2}{(\epsilon_2 - \epsilon_3)(\epsilon_1 - \epsilon_2)} \tag{24}$$

$$A_3' = \frac{[S(\epsilon_1 + \epsilon_2) + T\epsilon_1\epsilon_2 + C]\epsilon_3^3 - a_5 + a_4\epsilon_3 - a_3\epsilon_3^2}{(\epsilon_3 - \epsilon_1)(\epsilon_2 - \epsilon_3)} \tag{25}$$

$$y = a_0 + a_1 k + a_2 k^2 + \sum_{p=1}^{n}\left[a_{p+3}\frac{1}{k^p} + b_p\frac{y}{k^p}\right] \tag{27}$$

To perform a least squares fit on this form, we need to find the minimum of

$$\sum_j w_j (y_j - \widehat{y_j})\overline{(y_j - \widehat{y_j})} =$$

$$\sum_j w_j \left(y_j - a_0 - a_1 k_j - a_2 k_j^2 - \sum_{p=1}^{n}\left[a_{p+3}\frac{1}{k_j^p} + b_p\frac{y}{k_j^p}\right]\right) \times$$

$$\overline{\left(y_j - a_0 - a_1 k_j - a_2 k_j^2 - \sum_{p=1}^{n}\left[a_{p+3}\frac{1}{k_j^p} + b_p\frac{y}{k_j^p}\right]\right)} =$$

$$\sum_j w_j \left(y_j - a_0 - a_1 k_j - a_2 k_j^2 - \sum_{p=1}^{n}\left[a_{p+3}\frac{1}{k_j^p} + b_p\frac{y}{k_j^p}\right]\right) \times$$

$$\left(\overline{y_j} - \overline{a_0} - \overline{a_1} k_j - \overline{a_2} k_j^2 - \sum_{p=1}^{n}\left[\overline{a_{p+3}}\frac{1}{k_j^p} + b_p\frac{\overline{y}}{k_j^p}\right]\right)$$

Here $a_0, ..., a_{n+3}$, and y_j are the complex quantities, the remainder all being real variables and constants. $b_1 ... b_n$ *must* be maintained as real because all the b_p are additive and multiplicative combinations of the ϵ_q. Due to the mixture of real and imaginary coefficients, it is expedient to expand all the coefficients as the sums of real and imaginary parts.

$$\sum_j w_j \left(y_j - a_{\mathrm{R}0} - i\,a_{I0} - a_{R1}k_j - i\,a_{I1}k_j - a_{R2}k_j^2 - i\,a_{I2}k_j^2 - \sum_{p=1}^{n}\left[a_{Rp+3}\tfrac{1}{k_j^p} + i\,a_{Ip+3}\tfrac{1}{k_j^p} + b_p\tfrac{y_j}{k_j^p}\right]\right)$$
$$\times \left(\overline{y_j} - a_{\mathrm{R}0} + i\,a_{I0} - a_{R1}k_j + i\,a_{I1}k_j - a_{R2}k_j^2 + i\,a_{I2}k_j^2 - \sum_{p=1}^{n}\left[a_{Rp+3}\tfrac{1}{k_j^p} - i\,a_{Ip+3}\tfrac{1}{k_j^p} + b_p\tfrac{\overline{y_j}}{k_j^p}\right]\right)$$

We must successively take the derivative of this sum with respect to $a_{R0}, a_{I0}, ..., a_{Rn+3}, a_{In+3}$ and $b_1...b_n$, set each expression to zero, and solve to find the minimum. Using the notation $y_{Rj} \equiv Real(y_j)$ and $y_{Ij} \equiv Imaginary(y_j)$, first work the complex a_q's for $q = 0...2$ which are the terms outside the inner sum.

$$\frac{\partial}{\partial a_{Rq}} \sum_j w_j (...)(...) = 0 =$$
$$\sum_j w_j \Big(-y_j k_j^q + a_{\mathrm{R}0}k_j^q + i\,a_{I0}k_j^q + a_{R1}k_j^{q+1} + i\,a_{I1}k_j{}^{q+1} + a_{R2}k_j^{q+2} + i\,a_{I2}k_j^{q+2}$$
$$+ \sum_{p=1}^{n}\left[a_{Rp+3}k_j^{q-p} + i\,a_{Ip+3}k_j^{q-p} + b_p y_j k_j^{q-p}\right]\Big)$$
$$+ w_j \Big(-\overline{y_j}k_j^q + a_{\mathrm{R}0}k_j^q - i\,a_{I0}k_j^q + a_{R1}k_j^{q+1} - i\,a_{I1}k_j^{q+1} + a_{R2}k_j^{q+2} - i\,a_{I2}k_j^{q+2}$$
$$+ \sum_{p=1}^{n}\left[a_{Rp+3}k_j^{q-p} - i\,a_{Ip+3}k_j^{q-p} + b_p\overline{y_j}k_j^{q-p}\right]\Big) =$$

$$\sum_j w_j \Big(-(y_j + \overline{y_j})k_j^q + 2a_{\mathrm{R}0}k_j^q + 2a_{R1}k_j^{q+1} + 2a_{R2}k_j^{q+2}$$
$$+ \sum_{p=1}^{n}\left[2a_{Rp+3}k_j^{q-p} + b_p(y_j + \overline{y_j})k_j^{q-p}\right]\Big) = 0$$

$$\sum_j w_j \Big(-y_{Rj}k_j^q + a_{\mathrm{R}0}k_j^q + a_{R1}k_j^{q+1} + a_{R2}k_j^{q+2}$$
$$+ \sum_{p=1}^{n}\left[a_{Rp+3}k_j^{q-p} + b_p y_{Rj}k_j^{q-p}\right]\Big) = 0 \qquad (28)$$

$$\frac{\partial}{\partial a_{Iq}} \sum_j w_j (...)(...) = 0 =$$
$$\sum_j w_j \Big(i\,y_j k_j^q - i\,a_{\mathrm{R}0}k_j^q + a_{I0}k_j^q - i\,a_{R1}k_j^{q+1} + a_{I1}k_j{}^{q+1} - i\,a_{R2}k_j^{q+2} + a_{I2}k_j^{q+2}$$
$$- \sum_{p=1}^{n}\left[i\,a_{Rp+3}k_j^{q-p} - a_{Ip+3}k_j^{q-p} + i\,b_p y_j k_j^{q-p}\right]\Big)$$

$$+ w_j\Big(- i\,\overline{y_j}k_j^q + i\,a_{\mathrm{R}0}k_j^q + a_{I0}k_j^q + i\,a_{R1}k_j^{q+1} + a_{I1}k_j^{q+1} + i\,a_{R2}k_j^{q+2} + a_{I2}k_j^{q+2}$$

$$+ \sum_{p=1}^{n}\big[i\,a_{Rp+3}k_j^{q-p} + a_{Ip+3}k_j^{q-p} + i\,b_p\overline{y_j}k_j^{q-p}\,\big]\Big) =$$

$$\sum_j w_j\Big(i\,(y_j - \overline{y_j})k_j^q + 2a_{I0}k_j^q + 2a_{I1}{k_j}^{q+1} + 2a_{I2}k_j^{q+2}$$

$$+ \sum_{p=1}^{n}\big[\,a_{Ip+3}k_j^{q-p} - i\,b_p(y_j - \overline{y_j})k_j^{q-p}\big]\Big) =$$

$$\sum_j w_j\Big(- y_{Ij}k_j^q + \; a_{I0}k_j^q + \; a_{I1}{k_j}^{q+1} + \; a_{I2}k_j^{q+2}$$

$$+ \sum_{p=1}^{n}\big[\,a_{Ip+3}k_j^{q-p} + \; b_p y_{Ij}k_j^{q-p}\big]\Big) = 0 \tag{29}$$

Next, work the complex a_q's for $q = 3 \ldots 2 + n$, which are the a coefficient terms inside the inner sum.

$$\frac{\partial}{\partial a_{Rq}}\sum_j w_j(\ldots)(\ldots) = 0 =$$

$$\sum_j w_j\Big(- y_jk_j^{2-q} + a_{\mathrm{R}0}k_j^{2-q} + i\,a_{I0}k_j^{2-q} + a_{R1}k_j^{3-q} + i\,a_{I1}k_j^{3-q} + a_{R2}k_j^{4-q} + i\,a_{I2}k_j^{4-q}$$

$$+ \sum_{p=1}^{n}\Big[a_{Rp+3}k_j^{2-q-p} + i\,a_{Ip+3}k_j^{2-q-p} + b_py_jk_j^{2-q-p}\Big]\Big) \; + w_j\Big(- \overline{y_j}k_j^{2-q} + a_{\mathrm{R}0}k_j^{2-q} - i$$

$$a_{I0}k_j^{2-q} + a_{R1}k_j^{3-q} - i\,a_{I1}k_j^{3-q} + a_{R2}k_j^{4-q} - i$$

$$a_{I2}k_j^{4-q} + \sum_{p=1}^{n}\Big[a_{Rp+3}k_j^{2-q-p} - i\,a_{Ip+3}k_j^{2-q-p} + b_p\overline{y_j}k_j^{2-q-p}\Big]\Big) =$$

$$\sum_j w_j\Big(- (y_j + \overline{y_j})k_j^{2-q} + 2a_{\mathrm{R}0}k_j^{2-q} + 2a_{R1}k_j^{3-q} + 2a_{R2}k_j^{4-q}$$

$$+ \sum_{p=1}^{n}\Big[2a_{Rp+3}k_j^{2-q-p} + b_p(y_j + \overline{y_j})k_j^{2-q-p}\Big]\Big) \; =$$

$$\sum_j w_j\Big(- y_{Rj}k_j^{2-q} + a_{\mathrm{R}0}k_j^{2-q} + a_{R1}k_j^{3-q} + a_{R2}k_j^{4-q}$$

$$+ \sum_{p=1}^{n}\Big[a_{Rp+3}k_j^{2-q-p} + b_p y_{Rj}k_j^{2-q-p}\Big]\Big) \ = 0 \tag{30}$$

$\frac{\partial}{\partial a_{Iq}}\sum_j w_j(...)(...) = 0 =$
$\sum_j w_j\Big(+ i\, y_j k_j^{2-q} - i\, a_{\mathrm{R}0}k_j^{2-q} + a_{I0}k_j^{2-q} - i\, a_{R1}k_j^{3-q} + a_{I1}k_j^{3-q} - i\, a_{R2}k_j^{4-q} + a_{I2}k_j^{4-q}$
$- \sum_{p=1}^{n}\Big[i\, a_{Rp+3}k_j^{2-q-p} - a_{Ip+3}k_j^{2-q-p} + i\, b_p y_j k_j^{2-q-p}\Big]\Big) \ + w_j\Big(- i\,\overline{y_j}k_j^{2-q} + i$
$a_{\mathrm{R}0}k_j^{2-q} + a_{I0}k_j^{2-q} + i\, a_{R1}k_j^{3-q} + a_{I1}k_j^{3-q} + i$
$a_{R2}k_j^{4-q} + a_{I2}k_j^{4-q} + \sum_{p=1}^{n}\Big[i\, a_{Rp+3}k_j^{2-q-p} + a_{Ip+3}k_j^{2-q-p} + i\, b_p\overline{y_j}k_j^{2-q-p}\Big]\Big) =$

$$\sum_j w_j\Big(+ i\,(y_j - \overline{y_j})k_j^{2-q} + 2a_{I0}k_j^{2-q} + 2a_{I1}k_j^{3-q} + 2a_{I2}k_j^{4-q}$$

$$- \sum_{p=1}^{n}\Big[- 2a_{Ip+3}k_j^{2-q-p} + i\, b_p(y_j - \overline{y_j})k_j^{2-q-p}\Big]\Big) \ =$$

$$\sum_j w_j\Big(- y_{Ij}k_j^{2-q} + 2a_{I0}k_j^{2-q} + 2a_{I1}k_j^{3-q} + 2a_{I2}k_j^{4-q}$$

$$+ \sum_{p=1}^{n}\Big[2a_{Ip+3}k_j^{2-q-p} + \ b_p y_{Ij}k_j^{2-q-p}\Big]\Big) \ = 0 \tag{31}$$

Finally we need to work the real b_q's for $q = 1...n$.

$\frac{\partial}{\partial \mathrm{b}_q}\sum_j w_j(...)(...) = 0 =$
$\sum_j w_j\Big(- \overline{y_j}y_j k_j^{-q} + a_{\mathrm{R}0}\overline{y_j}k_j^{-q} + i\, a_{I0}\overline{y_j}k_j^{-q} + a_{R1}\overline{y_j}k_j^{1-q} + i\, a_{I1}\overline{y_j}k_j^{1-q} + a_{R2}\overline{y_j}k_j^{2-q} + i$
$a_{I2}\overline{y_j}k_j^{2-q} + \sum_{p=1}^{n}\left[a_{Rp+3}\overline{y_j}k_j^{-q-p} + i\, a_{Ip+3}\overline{y_j}k_j^{-q-p} + b_p\overline{y_j}y_j k_j^{-q-p}\right]\Big) +$
$w_j\Big(- y_j\overline{y_j}k_j^{-q} + a_{\mathrm{R}0}y_j k_j^{-q} - i\, a_{I0}y_j k_j^{-q} + a_{R1}y_j k_j^{1-q} - i\, a_{I1}y_j k_j^{1-q} + a_{R2}y_j k_j^{2-q} - i$
$a_{I2}y_j k_j^{2-q} + \sum_{p=1}^{n}\left[a_{Rp+3}y_j k_j^{-q-p} - i\, a_{Ip+3}y_j k_j^{-q-p} + b_p y_j\overline{y_j}k_j^{-q-p}\right]\Big) = 0$

$$\sum_j w_j \Big(-|y_j|^2 k_j^{-q} + a_{\mathrm{R}0} y_{Rj} k_j^{-q} + a_{I0} y_{Ij} k_j^{-q} + a_{R1} y_{Rj} k_j^{1-q} + a_{I1} y_{Ij} k_j^{1-q} + a_{R2} y_{Rj} k_j^{2-q}$$

$$+ a_{I2} y_{Ij} k_j^{2-q} + \sum_{p=1}^{n} \big[a_{Rp+3} y_{Rj} k_j^{-q-p} + a_{Ip+3} y_{Ij} k_j^{-q-p} + b_p |y_j|^2 k_j^{-q-p} \big] \Big) = 0 \quad (32)$$

In matrix form, these equations can be expressed as:

$$M\alpha = \beta$$

where α is the vector of coefficients and β is the vector of dependent variables, where each element represents a sum over j.

$$\alpha = \begin{bmatrix} a_{R0} \\ a_{I0} \\ a_{R1} \\ a_{I1} \\ a_{R2} \\ a_{I2} \\ a_{R3} \\ a_{I3} \\ \vdots \\ a_{Rn+2} \\ a_{In+2} \\ b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad \beta = \begin{bmatrix} w_j y_{Rj} \\ w_j y_{Ij} \\ w_j y_{Rj} k_j \\ w_j y_{Ij} k_j \\ w_j y_{Rj} k_j^2 \\ w_j y_{Ij} k_j^2 \\ w_j y_{Rj} k_j^{-1} \\ w_j y_{Ij} k_j^{-1} \\ \vdots \\ w_j y_{Rj} k_j^{-n} \\ w_j y_{Ij} k_j^{-n} \\ w_j |y_j|^2 k_j^{-1} \\ \vdots \\ w_j |y_j|^2 k_j^{-n} \end{bmatrix}$$

In the matrix M below, each element represents a represents a sum over j and also carries an implicit w_j, the weighting factor that is common to all elements. For example, in the representation below $M_{1,1} = \sum_j w_j$ and the last element of the first row $M_{1,2n+3} = \sum_j w_j Real(y_j) k^{-n}$.

$$
\begin{bmatrix}
1 & 0 & k & 0 & k^2 & 0 & k^{-1} & 0 & \dots & k^{-n} & 0 & y_R k^{-1} & \dots & y_R k^{-n} \\
0 & 1 & 0 & k & 0 & k^2 & 0 & k^{-1} & \dots & 0 & k^{-n} & y_I k^{-1} & \dots & y_I k^{-n} \\
k & 0 & k^2 & 0 & k^3 & 0 & 1 & 0 & \dots & k^{1-n} & 0 & y_R & \dots & y_R k^{1-n} \\
0 & k & 0 & k^2 & 0 & k^3 & 0 & 1 & \dots & 0 & k^{1-n} & y_I & \dots & y_I k^{1-n} \\
k^2 & 0 & k^3 & 0 & k^4 & 0 & k & 0 & \dots & k^{2-n} & 0 & y_R k & \dots & y_R k^{2-n} \\
0 & k^2 & 0 & k^3 & 0 & k^4 & 0 & k & \dots & 0 & k^{2-n} & y_I k & \dots & y_I k^{2-n} \\
k^{-1} & 0 & 1 & 0 & k & 0 & k^{-2} & 0 & \dots & k^{-1-n} & 0 & y_R k^{-2} & \dots & y_R k^{-1-n} \\
0 & k^{-1} & 0 & 1 & 0 & k & 0 & k^{-2} & \dots & 0 & k^{-1-n} & y_I k^{-2} & \dots & y_I k^{-1-n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
k^{-n} & 0 & k^{1-n} & 0 & k^{2-n} & 0 & k^{-1-n} & 0 & \dots & k^{-2n} & 0 & y_R k^{-1-n} & \dots & y_R k^{-2n} \\
0 & k^{-n} & 0 & k^{1-n} & 0 & k^{2-n} & 0 & k^{-1-n} & \dots & 0 & 0 & y_I k^{-1-n} & \dots & y_I k^{-2n} \\
y_R k^{-1} & y_I k^{-1} & y_R & y_I & y_R k & y_I k & y_R k^{-2} & y_I k^{-2} & \dots & y_R k^{-1-n} & y_I k^{-1-n} & |y|^2 k^{-2} & \dots & |y|^2 k^{-1-n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_R k^{-n} & y_I k^{-n} & y_R k^{1-n} & y_I k^{1-n} & y_R k^{2-n} & y_I k^{2-n} & y_R k^{-1-n} & y_I k^{-1-n} & \dots & y_R k^{-2n} & y_I k^{-2n} & |y|^2 k^{-1-n} & \dots & |y|^2 k^{-2n}
\end{bmatrix}
$$

As before, the coefficients α can be obtained by solving $M\alpha = \beta$, which can be done by inverting the matrix $\alpha = M^{-1}\beta$ or by other means. The frequencies and amplitudes are then obtained from the linear coefficients as shown above.

Preliminary results

This method was programmed and has given highly accurate results using test cases - in line with those for single frequencies tabulated in Supplement 1. In fact, it was possible to separate sets of two and three precise frequency signals which were all between two successive frequencies of the FFT being used. Although the treatment of three adjacent frequencies is a convenient and practical place to stop, there is no reason why even more adjacent signals could not be treated.

At this point it does appear important to determine how many signals are present in an interval and to perform the corresponding analysis in order to obtain maximum precision. This may be done before the fact, by examining the patters present in the Fourier coefficients, or after the fact, by examining the results of the fits after they are performed. The next step in development will be defining an efficient procedure to determine this.